Boundary Value Problems for the Quaternionic Hermitian System in $\mathbb{R}^{4n}$

Ricardo Abreu-Blaya Juan Bory-Reyes Fred Brackx
Hennie De Schepper Frank Sommen

Abstract

In this paper boundary value problems for quaternionic Hermitian monogenic functions are presented using a circulant matrix approach.

Keywords. Quaternionic Clifford analysis, Cauchy integral formula.
Mathematics Subject Classification (2000). 30G35.

1 Introduction

Euclidean Clifford analysis is a higher dimensional function theory offering a refinement of classical harmonic analysis. The theory is centred around the concept of monogenic functions, i.e. null solutions of a first order vector valued rotation invariant differential operator, called Dirac operator, which factorizes the Laplacian; monogenic functions may thus also be seen as a generalization of holomorphic functions in the complex plane. Its roots go back to the 1930's. For more details on this function theory we refer e.g. to the standard references [5, 14, 16, 17, 18].

More recently Hermitian Clifford analysis emerged as a refinement of the Euclidean setting for the case of $\mathbb{R}^{2n}$. Here, Hermitian monogenic functions are considered, i.e. functions taking values either in a complex Clifford algebra or in complex spinor space, and being simultaneous null solutions of two complex Hermitian Dirac operators, which are invariant under the action of the unitary group. For the systematic development of this function theory we refer e.g. to [6, 7, 9].

In the papers [12, 13, 15, 19], the Hermitian Clifford analysis setting was further refined by considering functions on $\mathbb{R}^{4n}$ with values in a quaternionic Clifford algebra, being simultaneous null solutions of four mutually related quaternionic Dirac operators, which are invariant under the action of the symplectic group. In [3], Borel–Pompeiu and Cauchy integral formulas are established in this setting, by following a (4×4) circulant matrix approach, similar in spirit to the circulant (2×2) matrix approach introduced in [10] within the complex Hermitian Clifford case.

Subsequently, in [4] a quaternionic Hermitian Cauchy integral is introduced, as well as its boundary limit values, leading to the definition of a matrix quaternionic Hermitian Hilbert transform. These operators provide a useful tool for studying boundary value problems for the quaternionic Hermitian system. This is precisely the main objective of the present paper. The main problems that we address are the problem of finding a quaternionic Hermitian monogenic function with a given jump over a given surface of $\mathbb{R}^{4n}$ as well as problems of Dirichlet type for the quaternionic Hermitian system. Finally, we also prove an equivalence between both sided quaternionic Hermitian monogenicity and a certain integral conservation law.

2 Preliminaries

Let $(e_1, \ldots, e_m)$ be an orthonormal basis of Euclidean space $\mathbb{R}^m$ and consider the real Clifford algebra $\mathbb{R}_{0,m}$ constructed over $\mathbb{R}^m$. The non-commutative multiplication in $\mathbb{R}_m$ is governed by the rules:

$$\begin{aligned} e_\ell^2 &= -1, \qquad \ell = 1, \ldots, m \\ e_\ell e_k + e_k e_\ell &= 0, \qquad \ell \neq k \end{aligned}$$

In $\mathbb{R}_m$ one can consider the following automorphisms:

(i) the conjugation $\overline{e}_\ell = -e_\ell$ and for any $a, b \in \mathbb{R}_m$, $\overline{ab} = \overline{b}\,\overline{a}$

(ii) the main involution $\tilde{e}_\ell = -e_\ell$ and for any $a, b \in \mathbb{R}_m$, $\widetilde{ab} = \tilde{a}\tilde{b}$

In particular we consider the skew–field of quaternions $\mathbb{H}$ whose elements will be denoted by $q = x_0 + ix_1 + jx_2 + kx_3$ with $i^2 = j^2 = k^2 = -1$ and $ij = -ji = k$. Clearly $\mathbb{H}$ may be identified with the Clifford algebra $\mathbb{R}_{0,2}$ making the identifications $i \leftrightarrow e_1$, $j \leftrightarrow e_2$ and $k \leftrightarrow e_1e_2$. The automorphisms (i) and (ii) then respectively lead to the $\mathbb{H}$–conjugation

$$\overline{q} = x_0 - ix_1 - jx_2 - kx_3$$

and to the main $\mathbb{H}$–involution

$$q^\gamma \equiv \tilde{q} = x_0 - ix_1 - jx_2 + kx_3$$

However, it is quite natural to introduce two more $\mathbb{H}$–involutions defined by

$$q^\alpha = x_0 + ix_1 - jx_2 - kx_3, \quad q^\beta = x_0 - ix_1 + jx_2 - kx_3$$

Definition 1 ([19]) *The quaternionic Witt basis of* $\mathbb{H}_m = \mathbb{H} \otimes_{\mathbb{R}} \mathbb{R}_m$, $m = 4n$, *is given by* $\{f_\ell, f_\ell^\alpha, f_\ell^\beta, f_\ell^\gamma\}$, $\ell = 1, \ldots, n$, *where*

$$\begin{aligned} f_\ell &= e_{1+4(\ell-1)} - ie_{2+4(\ell-1)} - je_{3+4(\ell-1)} - ke_{4+4(\ell-1)} \\ f_\ell^\alpha &= e_{1+4(\ell-1)} - ie_{2+4(\ell-1)} + je_{3+4(\ell-1)} + ke_{4+4(\ell-1)} \\ f_\ell^\beta &= e_{1+4(\ell-1)} + ie_{2+4(\ell-1)} - je_{3+4(\ell-1)} + ke_{4+4(\ell-1)} \\ f_\ell^\gamma &= e_{1+4(\ell-1)} + ie_{2+4(\ell-1)} + je_{3+4(\ell-1)} - ke_{4+4(\ell-1)} \end{aligned}$$

We will consider the Clifford vectors

$$\begin{aligned}
\underline{X} = \underline{X}_0 &= \sum_{\ell=1}^{n}(e_{4\ell-3}x_{4\ell-3} + e_{4\ell-2}x_{4\ell-2} + e_{4\ell-1}x_{4\ell-1} + e_{4\ell}x_{4\ell}) \\
\underline{X}_1 &= \sum_{\ell=1}^{n}(e_{4\ell-3}x_{4\ell-2} - e_{4\ell-2}x_{4\ell-3} - e_{4\ell-1}x_{4\ell} + e_{4\ell}x_{4\ell-1}) \\
\underline{X}_2 &= \sum_{\ell=1}^{n}(e_{4\ell-3}x_{4\ell-1} + e_{4\ell-2}x_{4\ell} - e_{4\ell-1}x_{4\ell-3} - e_{4\ell}x_{4\ell-2}) \\
\underline{X}_3 &= \sum_{\ell=1}^{n}(e_{4\ell-3}x_{4\ell} - e_{4\ell-2}x_{4\ell-1} + e_{4\ell-1}x_{4\ell-2} - e_{4\ell}x_{4\ell-3})
\end{aligned}$$

for which $\underline{X}_r^2 = -|\underline{X}_0|^2$, while $\underline{X}_r\underline{X}_s + \underline{X}_s\underline{X}_r = 0$, $r \neq s$, $r, s = 0, \ldots, 3$. The corresponding Dirac operators are denoted by $\partial_{\underline{X}} = \partial_{\underline{X}_0}$, $\partial_{\underline{X}_1}$, $\partial_{\underline{X}_2}$ and $\partial_{\underline{X}_3}$. Here we have $\partial^2_{\underline{X}_r} = -\Delta_{4n}$, with Δ_{4n} the Laplacian in $\mathbb{R}^{4n}$, and $\partial_{\underline{X}_r}\partial_{\underline{X}_s} + \partial_{\underline{X}_s}\partial_{\underline{X}_r} = 0$, $r \neq s$, $r, s = 0, \ldots, 3$. Next the quaternionic Hermitian variables are introduced:

$$\begin{aligned}
\underline{Z} = \underline{Z}_0 &= \underline{X}_0 + i\underline{X}_1 + j\underline{X}_2 + k\underline{X}_3 \\
\underline{Z}_1 &= \underline{X}_0 + i\underline{X}_1 - j\underline{X}_2 - k\underline{X}_3 \\
\underline{Z}_2 &= \underline{X}_0 - i\underline{X}_1 + j\underline{X}_2 - k\underline{X}_3 \\
\underline{Z}_3 &= \underline{X}_0 - i\underline{X}_1 - j\underline{X}_2 + k\underline{X}_3
\end{aligned}$$

for which $\underline{Z}_0\underline{Z}_0^\dagger + \underline{Z}_1\underline{Z}_1^\dagger + \underline{Z}_2\underline{Z}_2^\dagger + \underline{Z}_3\underline{Z}_3^\dagger = 16|\underline{X}|^2$, the symbol † denoting Hermitian quaternionic conjugation, defined as the composition of $\mathbb{H}$–conjugation and Clifford conjugation in $\mathbb{R}_{0,m}$, i.e. $\lambda^\dagger = \sum_A \overline{e_A}\,\overline{\lambda_A}$. The Hermitian Dirac operators are

$$\begin{aligned}
\partial_{\underline{Z}_0} &= \frac{1}{16}(\partial_{\underline{X}_0} + i\partial_{\underline{X}_1} + j\partial_{\underline{X}_2} + k\partial_{\underline{X}_3}) \\
\partial_{\underline{Z}_1} &= \frac{1}{16}(\partial_{\underline{X}_0} + i\partial_{\underline{X}_1} - j\partial_{\underline{X}_2} - k\partial_{\underline{X}_3}) \\
\partial_{\underline{Z}_2} &= \frac{1}{16}(\partial_{\underline{X}_0} - i\partial_{\underline{X}_1} + j\partial_{\underline{X}_2} - k\partial_{\underline{X}_3}) \\
\partial_{\underline{Z}_3} &= \frac{1}{16}(\partial_{\underline{X}_0} - i\partial_{\underline{X}_1} - j\partial_{\underline{X}_2} + k\partial_{\underline{X}_3})
\end{aligned}$$

for which $\Delta_{4n} = 16(\partial_{\underline{Z}_0}\partial^\dagger_{\underline{Z}_0} + \partial_{\underline{Z}_1}\partial^\dagger_{\underline{Z}_1} + \partial_{\underline{Z}_2}\partial^\dagger_{\underline{Z}_2} + \partial_{\underline{Z}_3}\partial^\dagger_{\underline{Z}_3})$.

Definition 2 *(see [19]) Let Ω be an open set in $\mathbb{R}^{4n}$. A continuously differentiable function $f : \Omega \mapsto \mathbb{H}_{4n}$ is said to be (left) q–Hermitian monogenic in Ω (or q–monogenic for short) iff it satisfies in Ω the system $\partial_{\underline{Z}_0} f = \partial_{\underline{Z}_1} f = \partial_{\underline{Z}_2} f = \partial_{\underline{Z}_3} f = 0$, or, equivalently, the system $\partial_{\underline{X}_0} f = \partial_{\underline{X}_1} f = \partial_{\underline{X}_2} f = \partial_{\underline{X}_3} f = 0$.*

Similarly right q–monogenicity is defined. Left and right q–monogenic functions are called two-sided q–monogenic. A q–monogenic function in Ω is monogenic, and thus harmonic in Ω. Note that Definition 2 was proven in [12] to be equivalent to the system introduced in [15] by group invariance considerations.

The fundamental solutions of the Dirac operators $\partial_{\underline{X}_r}$, $r = 0, \ldots, 3$, i.e. the Euclidean Cauchy kernels, are respectively given by

$$E_r(\underline{X}) = -\frac{1}{a_{4n}} \frac{\underline{X}_r}{|\underline{X}|^{4n}}, \qquad r = 0, \ldots, 3$$

with a_{4n} the area of the unit sphere S^{4n-1} in $\mathbb{R}^{4n}$. Explicitly, this means that $\partial_{\underline{X}_r} E_r(\underline{X}) = \delta(\underline{X})$, $r = 0, \ldots, 3$. Next we introduce the Hermitian Cauchy kernels:

$$\mathcal{E}_r(\underline{Z}) = \frac{1}{a_{4n}} \frac{\underline{Z}_r^\dagger}{|\underline{Z}|^{4n}}, \qquad r = 0, \ldots, 3$$

Note that $\mathcal{E}_r$ is not the fundamental solution of $\partial_{\underline{Z}_r}$. However, the following theorem holds, see [3].

Theorem 1 *Introducing the circulant* (4×4) *matrices*

$$\boldsymbol{\mathcal{D}} = \begin{pmatrix} \partial_{\underline{Z}_0} & \partial_{\underline{Z}_3} & \partial_{\underline{Z}_2} & \partial_{\underline{Z}_1} \\ \partial_{\underline{Z}_1} & \partial_{\underline{Z}_0} & \partial_{\underline{Z}_3} & \partial_{\underline{Z}_2} \\ \partial_{\underline{Z}_2} & \partial_{\underline{Z}_1} & \partial_{\underline{Z}_0} & \partial_{\underline{Z}_3} \\ \partial_{\underline{Z}_3} & \partial_{\underline{Z}_2} & \partial_{\underline{Z}_1} & \partial_{\underline{Z}_0} \end{pmatrix}, \boldsymbol{\mathcal{E}} = \begin{pmatrix} \mathcal{E}_0 & \mathcal{E}_3 & \mathcal{E}_2 & \mathcal{E}_1 \\ \mathcal{E}_1 & \mathcal{E}_0 & \mathcal{E}_3 & \mathcal{E}_2 \\ \mathcal{E}_2 & \mathcal{E}_1 & \mathcal{E}_0 & \mathcal{E}_3 \\ \mathcal{E}_3 & \mathcal{E}_2 & \mathcal{E}_1 & \mathcal{E}_0 \end{pmatrix}, \boldsymbol{\delta} = \begin{pmatrix} \delta & 0 & 0 & 0 \\ 0 & \delta & 0 & 0 \\ 0 & 0 & \delta & 0 \\ 0 & 0 & 0 & \delta \end{pmatrix}$$

one obtains that $\boldsymbol{\mathcal{D}}^T \boldsymbol{\mathcal{E}} = \boldsymbol{\mathcal{E}} \boldsymbol{\mathcal{D}}^T = \boldsymbol{\delta}$.

Thus, $\boldsymbol{\mathcal{E}}$ is a fundamental solution of $\boldsymbol{\mathcal{D}}$, in a matricial interpretation.

We associate, with functions g_0, g_1, g_2 and g_3 defined in $\Omega \subset \mathbb{R}^{4n}$ and taking values in $\mathbb{H}_{4n}$, the (4×4) circulant matrix function

$$\boldsymbol{G} \;=\; \begin{pmatrix} g_0 & g_3 & g_2 & g_1 \\ g_1 & g_0 & g_3 & g_2 \\ g_2 & g_1 & g_0 & g_3 \\ g_3 & g_2 & g_1 & g_0 \end{pmatrix} \equiv \operatorname{circ} \begin{pmatrix} g_0 \\ g_1 \\ g_2 \\ g_3 \end{pmatrix} \tag{1}$$

We say that $\boldsymbol{G}$ belongs to some class of functions if all its entries belong to that class. In particular, the spaces of k-times continuously differentiable, of α-Hölder continuous $(0 < \alpha \leq 1)$ and of p-integrable (4×4) circulant matrix functions on some suitable subset $\mathbf{E}$ of $\mathbb{R}^{4n}$ are respectively denoted by $\mathbf{C}^k(\mathbf{E})$, $\mathbf{C}^{0,\alpha}(\mathbf{E})$ and $\mathbf{L}^p(\mathbf{E})$. The corresponding spaces of $\mathbb{H}_{4n}$–valued functions are denoted by $C^k(\mathbf{E})$, $C^{0,\alpha}(\mathbf{E})$ and $L^p(\mathbf{E})$. Moreover, introducing the non–negative function $\|\boldsymbol{G}(\underline{X})\| = \max_{r=0,1,2,3} \{|g_r(\underline{X})|\}$, the classes $\mathbf{C}^{0,\alpha}(\mathbf{E})$ and $\mathbf{L}^p(\mathbf{E})$ may also be defined by means of the respective traditional conditions

$$\|\boldsymbol{G}\|_\alpha = \max_{\underline{X} \in \mathbf{E}} \|\boldsymbol{G}(\underline{X})\| + \sup_{\underline{X}, \underline{Y} \in \mathbf{E},\ \underline{X} \neq \underline{Y}} \frac{\|\boldsymbol{G}(\underline{X}) - \boldsymbol{G}(\underline{Y})\|}{|\underline{X} - \underline{Y}|^\alpha} \;<\; +\infty$$

and

$$\|\boldsymbol{G}\|_p = \left(\int_{\mathbf{E}} \|\boldsymbol{G}(\underline{X})\|^p \right)^{\frac{1}{p}} \; < \; +\infty$$

Definition 3 *The* (4×4) *circulant matrix function* $\boldsymbol{G}$ *is called (left)* $\mathbf{Q}$*–Hermitian monogenic in* Ω *(or Q–monogenic for short) iff* $\boldsymbol{\mathcal{D}}^T\boldsymbol{G} = \boldsymbol{O}$ *in* Ω*, where* $\boldsymbol{O}$ *denotes the matrix with zero entries.*

Similarly right $\mathbf{Q}$–monogenicity is defined by the system $\boldsymbol{G}\boldsymbol{\mathcal{D}}^T = \boldsymbol{O}$. Left and right $\mathbf{Q}$–monogenic matrix functions are called two-sided $\mathbf{Q}$–monogenic. An important special case concerns the diagonal matrix function $\boldsymbol{G}_0$, with $g_0 = g$ and $g_1 = g_2 = g_3 = 0$. Indeed, $\boldsymbol{G}_0$ is left (respectively right) $\mathbf{Q}$–monogenic iff the function g is left (respectively right) q–monogenic.

Now, let $\Omega^+ = \Omega$ be a bounded simply connected domain in $\mathbb{R}^{4n}$ with boundary $\Gamma = \partial\Omega$, and denote by Ω^- the complementary open domain $\mathbb{R}^{4n} \setminus (\Omega \cup \Gamma)$. We assume Γ to be a Liapunov surface. The unit normal vector on Γ at $\underline{X} \in \Gamma$ is given by

$$\underline{n}_0(\underline{X}) = \sum_{\ell=1}^{n} (e_{4\ell-3}n_{4\ell-3}(\underline{X}) + e_{4\ell-2}n_{4\ell-2}(\underline{X}) + e_{4\ell-1}n_{4\ell-1}(\underline{X}) + e_{4\ell}n_{4\ell}(\underline{X}))$$

and similarly as above, we also introduce the vectors $\underline{n}_1$, $\underline{n}_2$ and $\underline{n}_3$, giving rise in the usual way (up to a constant factor) to their Hermitian counterparts

$$\mathcal{N}_0 = \frac{1}{16}(\underline{n}_0 + i\underline{n}_1 + j\underline{n}_2 + k\underline{n}_3)$$

and $\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3$, as well as to the circulant matrix $\boldsymbol{\mathcal{N}}$. Then, in [3], the following Cauchy integral formulae were proven for $\mathbf{Q}$– monogenic matrix functions and for q–monogenic functions, respectively.

Theorem 2 (Q–Hermitian Cauchy integral formula) *If the matrix function* $\boldsymbol{G}$*, (1), is* $\mathbf{Q}$*–monogenic in* Ω *then*

$$\int_{\partial\Gamma} \boldsymbol{\mathcal{E}}(\underline{Z} - \underline{V})\boldsymbol{\mathcal{N}}^T(\underline{Z})\boldsymbol{G}(\underline{X})dS(\underline{X}) = \begin{cases} \boldsymbol{G}(\underline{Y}), & \underline{Y} \in \Gamma^+, \\ \boldsymbol{O}, & \underline{Y} \in \Gamma^-. \end{cases}$$

Theorem 3 (q–Hermitian Cauchy integral formula) *If the function* g *is* q*–monogenic in* Ω *then*

$$\int_{\partial\Gamma} \boldsymbol{\mathcal{E}}(\underline{Z} - \underline{V})\boldsymbol{\mathcal{N}}^T(\underline{Z})\boldsymbol{G}_0(\underline{X})dS(\underline{X}) = \begin{cases} \boldsymbol{G}_0(\underline{Y}), & \underline{Y} \in \Gamma^+, \\ \boldsymbol{O}, & \underline{Y} \in \Gamma^-, \end{cases}$$

where $\boldsymbol{G}_0$ *is the corresponding diagonal matrix.*

Next, in [4] a **Q**–Hermitian Cauchy transform was introduced, given by

$$\boldsymbol{\mathcal{C}}\left[\boldsymbol{G}\right](\underline{Y}) = \int_\Gamma \boldsymbol{\mathcal{E}}(\underline{Z}-\underline{V})\boldsymbol{\mathcal{N}}^T(\underline{Z})\boldsymbol{G}(\underline{X})dS(\underline{X}), \qquad \underline{Y}\notin\Gamma \tag{2}$$

for a matrix function $\boldsymbol{G}\in C(\Gamma)$, where $\underline{Z}$ and $\underline{V}$ denote the Hermitian versions of the Clifford vectors $\underline{X}$ and $\underline{Y}$, respectively. $\boldsymbol{\mathcal{C}}\left[\boldsymbol{G}\right]$ is a left **Q**–monogenic matrix function in $\mathbb{R}^{4n}\setminus\Gamma$, vanishing at infinity; in terms of the Euclidean Cauchy type integrals

$$C_{r,s}\, g(\underline{Y}) := \int_\Gamma E_r(\underline{X}-\underline{Y})\underline{n}_s(\underline{X})g(\underline{X})dS(\underline{X}), \qquad \underline{Y}\notin\Gamma$$

it reads as

$$\boldsymbol{\mathcal{C}}\left[\boldsymbol{G}\right] = \frac{1}{4}\,\mathrm{circ}\begin{pmatrix} C_{0,0}+C_{1,1}+C_{2,2}+C_{3,3} \\ C_{0,0}-C_{2,2}+j(C_{1,3}+C_{3,1}) \\ C_{0,0}-C_{1,1}+C_{2,2}-C_{3,3} \\ C_{0,0}-C_{2,2}-j(C_{1,3}+C_{3,1}) \end{pmatrix}[\boldsymbol{G}]$$

In particular, for the special case of the matrix $\boldsymbol{G}_0$, the action of $\boldsymbol{\mathcal{C}}$ is reduced to

$$\boldsymbol{\mathcal{C}}\left[\boldsymbol{G}_0\right] = \frac{1}{4}\,\mathrm{circ}\begin{pmatrix} C_{0,0}g+C_{1,1}g+C_{2,2}g+C_{3,3}g \\ C_{0,0}g-C_{2,2}g+j(C_{1,3}g+C_{3,1}g) \\ C_{0,0}g-C_{1,1}g+C_{2,2}g-C_{3,3}g \\ C_{0,0}g-C_{2,2}g-j(C_{1,3}g+C_{3,1}g) \end{pmatrix}$$

In general $\boldsymbol{\mathcal{C}}\left[\boldsymbol{G}_0\right]$ will not be a diagonal matrix, whence its entries will not be left q–monogenic functions. However $\boldsymbol{\mathcal{C}}\left[\boldsymbol{G}_0\right]$ does become diagonal if and only if

$$C_{0,0}g = C_{2,2}g, \quad C_{1,3}g = -C_{3,1}g, \quad 2C_{0,0}g = C_{1,1}g + C_{3,3}g \tag{3}$$

in which case we obtain

$$\boldsymbol{\mathcal{C}}\left[\boldsymbol{G}_0\right] = \mathrm{circ}\begin{pmatrix} C_{0,0}g \\ 0 \\ 0 \\ 0 \end{pmatrix} = \mathrm{circ}\begin{pmatrix} C_{2,2}g \\ 0 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{2}\,\mathrm{circ}\begin{pmatrix} C_{1,1}g+C_{3,3}g \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

The following Plemelj-Sokhotski formula, proven in [4], then asserts the existence of the continuous boundary limits of the **Q**-Hermitian Cauchy transform.

Theorem 4 *Let $\boldsymbol{G}\in\mathbf{C}^{0,\alpha}(\Gamma)$ $(0<\alpha\leq 1)$, then the continuous limit values of its **Q**-Hermitian Cauchy transform $\boldsymbol{\mathcal{C}}\left[\boldsymbol{G}\right]$ exist and are given by*

$$\boldsymbol{\mathcal{C}}^{\pm}\left[\boldsymbol{G}\right](\underline{U}) = \frac{1}{2}\left(\boldsymbol{\mathcal{H}}[\boldsymbol{G}](\underline{U}) \pm \boldsymbol{G}(\underline{U})\right), \qquad \underline{U}\in\Gamma.$$

Here we have introduced the matrix **Q**-Hermitian Hilbert operator

$$\boldsymbol{\mathcal{H}}[\boldsymbol{G}] \;=\; \frac{1}{4}\,\mathrm{circ}\begin{pmatrix} H_{0,0}+H_{1,1}+H_{2,2}+H_{3,3} \\ H_{0,0}-H_{2,2}+j(H_{1,3}+H_{3,1}) \\ H_{0,0}-H_{1,1}+H_{2,2}-H_{3,3} \\ H_{0,0}-H_{2,2}-j(H_{1,3}+H_{3,1}) \end{pmatrix}[\boldsymbol{G}]$$

where the singular integrals

$$H_{r,s}g(\underline{U}) = 2\int_{\Gamma} E_r(\underline{X}-\underline{U})\underline{n}_s(\underline{X})g(\underline{X})dS(\underline{X}), \qquad \underline{U}\in\Gamma$$

are Cauchy principal values. $\boldsymbol{\mathcal{H}}$ shows the following traditional properties, see [4].

Theorem 5 *One has*

(i) $\boldsymbol{\mathcal{H}}$ *is a bounded linear operator on* $\left(\mathbf{C}^{0,\alpha}(\Gamma), \|\bullet\|_{\alpha}\right)$ *(*$0<\alpha<1$*)*

(ii) $\boldsymbol{\mathcal{H}}$ *is an involution on* $\mathbf{C}^{0,\alpha}(\Gamma)$ *(*$0<\alpha<1$*).*

Similar results may be obtained for right-hand versions of the **Q**-Hermitian Cauchy and Hilbert transforms, by means of the alternative definitions

$$[\boldsymbol{G}]\boldsymbol{\mathcal{C}}\,(\underline{Y}) = \int_{\Gamma} \boldsymbol{G}(\underline{X})\boldsymbol{\mathcal{N}}^T(\underline{Z})\boldsymbol{\mathcal{E}}(\underline{Z}-\underline{V})dS(\underline{X}), \qquad \underline{Y}\notin\Gamma$$

and

$$[\boldsymbol{G}]\boldsymbol{\mathcal{H}} \;=\; [\boldsymbol{G}]\frac{1}{4}\,\mathrm{circ}\begin{pmatrix} H_{0,0}+H_{1,1}+H_{2,2}+H_{3,3} \\ H_{0,0}-H_{2,2}+j(H_{1,3}+H_{3,1}) \\ H_{0,0}-H_{1,1}+H_{2,2}-H_{3,3} \\ H_{0,0}-H_{2,2}-j(H_{1,3}+H_{3,1}) \end{pmatrix}$$

where

$$g\,H_{r,s}(\underline{U}) = 2\int_{\Gamma} g(\underline{X})\underline{n}_s(\underline{X})E_r(\underline{X}-\underline{U})dS(\underline{X}), \qquad \underline{U}\in\Gamma$$

3 Boundary value problems for Q–monogenic functions

In this section we study the so-called jump problem (reconstruction problem) for **Q**–monogenic functions, that is, we will investigate the problem of reconstructing a **Q**–monogenic matrix function $\boldsymbol{\Psi}$ in $\mathbb{R}^{4n}\setminus\Gamma$ vanishing at infinity and having a prescribed jump $\boldsymbol{G}$ across Γ, i.e.

$$\boldsymbol{\Psi}^+(\underline{\mathbf{U}}) \;-\; \boldsymbol{\Psi}^-(\underline{\mathbf{U}}) \;=\; \boldsymbol{G}(\underline{\mathbf{U}}), \qquad \underline{\mathbf{U}}\in\boldsymbol{\Gamma}. \tag{4}$$

First, it should be noted that if this problem has a solution, then it necesarily is unique. This assertion can be easily proven using the Painlevé and Liouville theorems in the Clifford analysis setting, see [1]. Next, under the condition that $\boldsymbol{G} \in \mathbf{C}^{0,\alpha}(\Gamma)$, Theorem 4 ensures the solvability of the jump problem (4), its unique solution being given by

$$\boldsymbol{\Psi}(\underline{\mathbf{Y}}) \;=\; \boldsymbol{\mathcal{C}}\,[\boldsymbol{G}](\underline{\mathbf{Y}}), \qquad \underline{\mathbf{Y}} \in \mathbb{R}^{\mathbf{4n}} \setminus \boldsymbol{\Gamma}.$$

Now consider the important special case of the matrix function $\boldsymbol{G}_0$. The reconstruction problem (4) then is strongly related to the jump problem for the involved q–monogenic function, as addressed in the following theorem.

Theorem 6 *For a function $g \in C^{0,\alpha}(\Gamma)$, the following statements are equivalent:*

(i) the jump problem

$$\psi^+(\underline{U}) - \psi^-(\underline{U}) = g(\underline{U}), \qquad \underline{U} \in \Gamma \tag{5}$$

is solvable in terms of q–monogenic functions;

(ii) g satisfies the relations (3);

(iii) g satisfies the relations $C_{0,0}g = C_{1,1}g = C_{2,2}g = C_{3,3}g$.

Proof

$(i) \rightarrow (ii)$

Associate to the function g the diagonal matrix function $\boldsymbol{G}_0$. Then $\boldsymbol{G}_0 \in \mathbf{C}^{0,\alpha}(\Gamma)$, and the jump problem (4) for $\boldsymbol{G}_0$ has the unique solution

$$\boldsymbol{\Psi}(\underline{\mathbf{Y}}) = \boldsymbol{\mathcal{C}}\,[\boldsymbol{G}_\mathbf{0}](\underline{\mathbf{Y}}), \qquad \underline{\mathbf{Y}} \in \mathbb{R}^{\mathbf{4n}} \setminus \boldsymbol{\Gamma}.$$

Let ψ be a solution of (5), then the circulant matrix

$$\boldsymbol{\Psi}(\underline{\mathbf{Y}}) = \operatorname{circ}\begin{pmatrix} \psi \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

is another solution of the jump problem (4) for $\boldsymbol{G}_0$, whence the uniqueness yields

$$\operatorname{circ}\begin{pmatrix} \psi \\ 0 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{4}\operatorname{circ}\begin{pmatrix} C_{0,0}g + C_{1,1}g + C_{2,2}g + C_{3,3}g \\ C_{0,0}g - C_{2,2}g + j(C_{1,3}g + C_{3,1}g) \\ C_{0,0}g - C_{1,1}g + C_{2,2}g - C_{3,3}g \\ C_{0,0}g - C_{2,2}g - j(C_{1,3}g + C_{3,1}g) \end{pmatrix}$$

implying (ii).

(ii) $\rightarrow$ *(iii)*
From the third relation in (3) we have $2\partial_{\underline{Y}_1} C_{0,0}g = \partial_{\underline{Y}_1} C_{3,3}g + \partial_{\underline{Y}_1} C_{1,1}g = \partial_{\underline{Y}_1} C_{3,3}$, and hence

$$\begin{aligned} 2\partial_{\underline{Y}_1} C_{0,0}g &= \int\limits_{\Gamma} \left(\partial_{\underline{Y}_1} E_3(\underline{X}-\underline{Y})\right) \underline{n}_3(\underline{X}) g(\underline{X}) dS(\underline{X}) \\ &= -\int\limits_{\Gamma} \left(\partial_{\underline{Y}_3} E_1(\underline{X}-\underline{Y})\right) \underline{n}_3(\underline{X}) g(\underline{X}) dS(\underline{X}) \\ &= -\partial_{\underline{Y}_3} C_{1,3}g \;=\; \partial_{\underline{Y}_3} C_{3,1}g \;=\; 0, \qquad \underline{Y} \notin \Gamma \end{aligned}$$

the latter following from the second relation in (3) and the $\partial_{\underline{Y}_3}$–monogenicity of $C_{3,1}g$. This fact means that $C_{0,0}g - C_{3,3}g$ is a $\partial_{\underline{Y}_1}$–monogenic function in $\mathbb{R}^{4n} \setminus \Gamma$. Moreover it has a null jump through Γ, whence it vanishes in the whole of $\mathbb{R}^{4n}$. We conclude that $C_{0,0}g = C_{3,3}g$. Similarly we arrive at $C_{0,0}g = C_{1,1}g$.

(iii) $\rightarrow$ *(i)*
It suffices to observe that, under the conditions stated, $C_{0,0}g$ is q–monogenic, whence it solves the jump problem (5). $\square$

For right q–monogenic functions the following analogue is obtained.

Theorem 7 *For a function $g \in C^{0,\alpha}(\Gamma)$, the following statements are equivalent:*

(i) the jump problem

$$\psi^+(\underline{U}) - \psi^-(\underline{U}) \;=\; g(\underline{U}), \qquad \underline{U} \in \Gamma \tag{6}$$

is solvable in terms of right q–monogenic functions;

(ii) g satisfies the relations

$$g\,C_{0,0} = g\,C_{2,2},\;\; g\,C_{1,3} = -g\,C_{3,1},\;\; 2g\,C_{0,0} = g\,C_{1,1} + g\,C_{3,3}$$

(iii) g satisfies the relations $g\,C_{0,0} = g\,C_{1,1} = g\,C_{2,2} = g\,C_{3,3}$.

The next result deals with the Dirichlet boundary value problem for **Q**–monogenic functions.

Theorem 8 *Let $\boldsymbol{G} \in \mathbf{C}^{0,\alpha}(\Gamma)$, then the following statements are equivalent:*

(i) The Dirichlet problem

$$\begin{aligned} \boldsymbol{\mathcal{D}}^T \mathbf{F} = \boldsymbol{O}\,(\textit{resp. } \mathbf{F}\boldsymbol{\mathcal{D}}^T = \boldsymbol{O}),\;\; &\textit{in } \Omega \\ \mathbf{F} = \boldsymbol{G},\;\; &\textit{on } \Gamma \end{aligned} \tag{7}$$

has a solution.

(ii) $\boldsymbol{\mathcal{H}}[\boldsymbol{G}] = \boldsymbol{G}$ (resp. $[\boldsymbol{G}]\boldsymbol{\mathcal{H}} = \boldsymbol{G}$)

Proof

We give the proof for the left–sided version of the theorem, the right–sided one being completely similar.

$(i) \to (ii)$

Let $\mathbf{F}$ be a solution of the Dirichlet problem (7). Then, by the $\mathbf{Q}$-Hermitian Cauchy formula, we have

$$\boldsymbol{\mathcal{C}}\left[\mathbf{F}\right](\underline{Y}) = \mathbf{F}(\underline{Y}), \qquad \underline{Y} \in \Omega$$

Taking limits as $\underline{Y} \to \underline{U} \in \Gamma$, (ii) follows in view of Theorem 4.

$(ii) \to (i)$

It suffices to observe that, under the condition (ii), $\mathbf{F} = \boldsymbol{\mathcal{C}}\left[\boldsymbol{G}\right]$ solves (7). $\square$

Theorem 9 *Let $g \in C^{0,\alpha}(\Gamma)$, then the following statements are equivalent:*

(i) The Dirichlet problem

$$\begin{aligned} \partial_{\underline{Z}_0} f = \partial_{\underline{Z}_1} f = \partial_{\underline{Z}_2} f = \partial_{\underline{Z}_3} f = 0, &\qquad \textit{in } \ \Omega \\ f = g, &\qquad \textit{on } \ \Gamma \end{aligned} \tag{8}$$

has a solution.

(ii) g satisfies the relations

$$H_{0,0}g = H_{2,2}g = g, \ H_{1,3}g = -H_{3,1}g, \ H_{1,1}g + H_{3,3}g = 2g$$

(iii) g satisfies the relations

$$H_{0,0}g = H_{1,1}g = H_{2,2}g = H_{3,3}g = g$$

Proof

$(i) \to (ii)$

From (i) we see that the matrix function

$$\mathbf{F}_0 = \ \mathrm{circ} \begin{pmatrix} f \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

is a solution of the Dirichlet problem

$$\begin{aligned} \boldsymbol{\mathcal{D}}^T\mathbf{F} = \boldsymbol{O}, &\ \text{ in } \Omega \\ \mathbf{F} = \boldsymbol{G}_0, &\ \text{ on } \Gamma \end{aligned}$$

whence by Theorem 8 we have that $\boldsymbol{\mathcal{H}}[\boldsymbol{G}_0] = \boldsymbol{G}_0$. The desired conclusion (ii) then directly follows by comparing the entries in the above equality.

(ii) $\rightarrow$ *(iii)*
From the condition $H_{0,0}g = H_{2,2}g = g$ it follows that $C^{\pm}_{0,0}\, g = C^{\pm}_{2,2}\, g$. Therefore, as $C_{0,0}\, g - C_{2,2}\, g$ is harmonic in $\Omega^{\pm}$ and $C^{\pm}_{0,0}\, g - C^{\pm}_{2,2}\, g|_{\Gamma} = 0$, we have $C_{0,0}\, g = C_{2,2}\, g$ in $\mathbb{R}^{4n} \setminus \Gamma$. Using the remaining conditions in (ii) and following a similar reasoning as above, we obtain that g satisfies the relations (3) and hence by Theorem 6 we have that $C_{0,0}g = C_{1,1}g = C_{2,2}g = C_{3,3}g$. Consequently, we obtain that $H_{0,0}g = H_{1,1}g = H_{2,2}g = H_{3,3}g = g$, as stated in (iii).

(iii) $\rightarrow$ *(i)*
The conditions $H_{0,0}g = H_{1,1}g = H_{2,2}g = H_{3,3}g = g$ imply the solvability of the Dirichlet problems

$$\begin{aligned} \partial_{\underline{X}_r} f &= 0, \text{ in } \Omega \\ f &= g, \text{ on } \Gamma \end{aligned} \tag{9}$$

where $r = 0, \ldots, 3$. Now, let f_0, f_1, f_2, f_3 be the respective solutions of (9), then these functions all are solutions of the classical Dirichlet problem

$$\begin{aligned} \Delta_{4n} f &= 0, \text{ in } \Omega \\ f &= g, \text{ on } \Gamma \end{aligned}$$

whence they coincide. The function $f = f_0 = f_1 = f_2 = f_3$ thus is q–monogenic and constitutes a solution of (8). $\square$

For right q–monogenic functions the following analogue is obtained.

Theorem 10 *Let $g \in C^{0,\alpha}(\Gamma)$, then the following statements are equivalent:*

(i) The Dirichlet problem

$$\begin{aligned} f\,\partial_{\underline{Z}_0} = f\,\partial_{\underline{Z}_1} = f\,\partial_{\underline{Z}_2} = f\,\partial_{\underline{Z}_3} &= 0, \qquad \text{in } \Omega \\ f &= g, \qquad \text{on } \Gamma \end{aligned} \tag{10}$$

has a solution.

(ii) g satisfies the relations

$$g\,H_{0,0} = g\,H_{2,2} = g,\; g\,H_{1,3} = -g\,H_{3,1},\; g\,H_{1,1} + g\,H_{3,3} = 2g$$

(iii) g satisfies the relations

$$g\,H_{0,0} = g\,H_{1,1} = g\,H_{2,2} = g\,H_{3,3}g = g$$

We now turn our attention towards establishing a connection between the two–sided $\mathbf{Q}$–monogenicity of a matrix function $\boldsymbol{G}$ and the matrix Hilbert transforms $\boldsymbol{\mathcal{H}}[\boldsymbol{G}|_{\Gamma}]$ and $[\boldsymbol{G}|_{\Gamma}]\boldsymbol{\mathcal{H}}$ of its trace on the boundary Γ.

Theorem 11 *Let $\boldsymbol{G} \in \mathbf{C}^{0,\alpha}(\Omega \cup \Gamma)$, such that $\boldsymbol{\mathcal{D}}^T \boldsymbol{G} = \boldsymbol{O}$ in Ω, then the following statements are equivalent:*

(i) $\boldsymbol{G}$ *is two–sided* $\mathbf{Q}$*–monogenic in* Ω.

(ii) $\boldsymbol{\mathcal{H}}[\boldsymbol{G}|_\Gamma] = [\boldsymbol{G}|_\Gamma]\boldsymbol{\mathcal{H}}$

Proof
Assume that, next to its already assumed left $\mathbf{Q}$–monogenicity, $\boldsymbol{G}$ also is right $\mathbf{Q}$–monogenic in Ω. Then by Theorem 8 it holds that

$$\boldsymbol{\mathcal{H}}[\boldsymbol{G}|_\Gamma] = \boldsymbol{G}|_\Gamma = [\boldsymbol{G}|_\Gamma]\boldsymbol{\mathcal{H}}$$

Conversely, suppose that $\boldsymbol{\mathcal{H}}[\boldsymbol{G}|_\Gamma] = [\boldsymbol{G}|_\Gamma]\boldsymbol{\mathcal{H}}$. By Theorem 4 and its right–handed version, we conclude that the corresponding left and right $\mathbf{Q}$–Hermitian Cauchy transform of $\boldsymbol{G}$, $\boldsymbol{\mathcal{C}}\,[\boldsymbol{G}]$ and $[\boldsymbol{G}]\,\boldsymbol{\mathcal{C}}$, have the same boundary values on Γ. This fact, together with their harmonicity, implies that

$$\boldsymbol{\mathcal{C}}\,[\boldsymbol{G}] = [\boldsymbol{G}]\,\boldsymbol{\mathcal{C}}$$

On the other hand, from the assumed left $\mathbf{Q}$–monogenicity of $\boldsymbol{G}$ we have $\boldsymbol{G} = \boldsymbol{\mathcal{C}}\,[\boldsymbol{G}]$ and hence

$$\boldsymbol{G} = \boldsymbol{\mathcal{C}}\,[\boldsymbol{G}] = [\boldsymbol{G}]\,\boldsymbol{\mathcal{C}}$$

which clearly forces $\boldsymbol{G}$ to be two-sided $\mathbf{Q}$–monogenic. $\square$

The following result illustrates the utility of the above theorem when considering q–monogenic functions.

Theorem 12 *Let $g \in C^{0,\alpha}(\Omega \cup \Gamma)$ be left q–monogenic in Ω, then the following statements are equivalent:*

(i) g *is two–sided* q*–monogenic in* Ω.

(ii) g *satisfies the relations*

$$\begin{aligned} H_{0,0}g = g\,H_{0,0},\ H_{2,2}g &= g\,H_{2,2} \\ H_{1,3}g + H_{3,1}g &= g\,H_{1,3} + g\,H_{3,1} \\ H_{1,1}g + H_{3,3}g &= g\,H_{1,1} + g\,H_{3,3} \end{aligned}$$

(iii) g *satisfies the relations*

$$H_{0,0}g = g\,H_{0,0},\ H_{1,1}g = g\,H_{1,1},\ H_{2,2}g = g\,H_{2,2},\ H_{3,3}g = g\,H_{3,3}$$

Proof

$(i) \leftrightarrow (ii)$
From (i) we see that the matrix function $\boldsymbol{G}_0$ corresponding to g is two–sided

$\mathbf{Q}$–monogenic in Ω, whence (ii) follows from Theorem 11(ii) applied to $\boldsymbol{G}_0$. Conversely, (ii) can be rewritten in the matricial form $\boldsymbol{\mathcal{H}}[\boldsymbol{G}_0|_\Gamma] = [\boldsymbol{G}_0|_\Gamma]\boldsymbol{\mathcal{H}}$, from which (i) follows by observing that the two–sided $\mathbf{Q}$–monogenicity of $\boldsymbol{G}_0$ implied by Theorem 11 is equivalent to the q–monogenicity of g.

$(i) \leftrightarrow (iii)$
It follows from (i) that g is two–sided monogenic w.r.t. $\partial_{\underline{X}_r}$, $r = 0, \ldots, 3$. We may then invoke [2, Theorem 3.2] in order to conclude that $H_{r,r}g = g\, H_{r,r}$, $r = 0, \ldots, 3$. Conversely, suppose that (iii) holds. Each of the conditions $H_{r,r}g = g\, H_{r,r}$, $r = 0, \ldots, 3$, implies the two–sided monogenicity of g in Ω w.r.t. $\partial_{\underline{X}_r}$, $r = 0, \ldots, 3$, see again [2, Theorem 3.2], whence g is two–sided q–monogenic in Ω. $\square$

4 Acknowledgments

Ricardo Abreu Blaya and Juan Bory Reyes wish to thank all members of the Department of Mathematical Analysis of Ghent University, where the paper was written, for the invitation and hospitality. They were supported respectively by the Research Council of Ghent University and by the Research Foundation - Flanders (FWO, project 31506208).

References

[1] R. Abreu Blaya, J. Bory Reyes, D. Peña Peña, Jump problem and removable singularities for monogenic functions, *J. Geom. Anal.* **17(1)** (2007), 1–13.

[2] R. Abreu Blaya, J. Bory-Reyes, R. Delanghe, F. Sommen, Cauchy integral decomposition of multi-vector valued functions on hypersurfaces, *Comput. Methods Funct. Theory* **5(1)** (2005), 111–135.

[3] R. Abreu Blaya, J. Bory Reyes, F. Brackx, H. De Schepper and F. Sommen, Cauchy Integral Formulae in Hermitian Quaternionic Clifford Analysis, accepted for publication in *Compl. Anal. Oper. Theory.*

[4] R. Abreu Blaya, J. Bory Reyes, F. Brackx, H. De Schepper and F. Sommen, Matrix Cauchy and Hilbert transforms in Hermitean quaternionic Clifford analysis, accepted for publication in *Comp. Var. Elliptic Equ.*

[5] F. Brackx, R. Delanghe, F. Sommen, *C*lifford Analysis, Research Notes in Mathematics **76**, Pitman, Boston, MA, 1982.

[6] F. Brackx, J. Bureš, H. De Schepper, D. Eelbode, F. Sommen, V. Souček, Fundaments of Hermitean Clifford analysis – Part I: Complex structure, *Complex Anal. Oper. Theory* **1(3)** (2007), 341–365.

[7] F. Brackx, J. Bureš, H. De Schepper, D. Eelbode, F. Sommen, V. Souček, Fundaments of Hermitean Clifford analysis – Part II: Splitting of h-monogenic equations, *Comp. Var. Elliptic Equ.* **52(10–11)** (2007), 1063–1079.

[8] F. Brackx, H. De Schepper, N. De Schepper, F. Sommen, Hermitean Clifford-Hermite wavelets: an alternative approach, *Bull. Belg. Math. Soc.–Simon Stevin* **15(1)** (2008), 87–107.

[9] F. Brackx, H. De Schepper, F. Sommen, The Hermitean Clifford analysis toolbox, *Adv. Appl. Cliff. Alg.* **18(3–4)** (2008), 451–487.

[10] F. Brackx, B. De Knock, H. De Schepper, A matrix Hilbert transform in Hermitean Clifford analysis, *J. Math. Anal. Appl.* **344** (2008), 1068–1078.

[11] F. Brackx, B. De Knock, H. De Schepper, F. Sommen, On Cauchy and Martinelli–Bochner integral formulae in Hermitean Clifford analysis, *Bull. Braz. Math. Soc.* **40(3)** (2009), 395–416.

[12] A. Damiano, D. Eelbode, I. Sabadini, Quaternionic Hermitian spinor systems and compatibility conditions, *Adv. Geom.* **11** (2011), 169-189.

[13] A. Damiano, D. Eelbode, I. Sabadini, Algebraic analysis of Hermitian monogenic functions, *C.R. Acad. Sci. Paris* Ser.I 346 (2008), 139-142.

[14] R. Delanghe, F. Sommen, V. Souček, *Clifford Algebra and Spinor–Valued Functions*, Kluwer Academic Publishers, Dordrecht, 1992.

[15] D. Eelbode, A Clifford algebraic framework for sp(m)-invariant differential operators, *Adv. Appl. Cliff. Algebras* **17** (2007), 635-649.

[16] J. Gilbert, M. Murray, *Clifford Algebras and Dirac Operators in Harmonic Analysis*, Cambridge University Press, Cambridge, 1991.

[17] K. Gürlebeck, K. Habetha, W. Sprössig, *Holomorphic functions in the plane and n-dimensional space*, translated from the 2006 German original, Birkhäuser Verlag, Basel, 2008.

[18] K. Gürlebeck, W. Sprössig, *Quaternionic and Clifford Calculus for Physicists and Engineers*, Wiley, Chichester, 1998.

[19] D. Peña-Peña, I. Sabadini, F. Sommen, Quaternionic Clifford analysis: the Hermitian setting, *Complex Anal. Oper. Theory* **1** (2007), 97–113.

[20] I. Sabadini, F. Sommen, Hermitian Clifford analysis and resolutions, *Math. Methods Appl. Sci.* **25(16–18)** (2002), 1395–1413.

R. Abreu Blaya
Facultad de Informática y Matemática, Universidad de Holguín, Holguín 80100, Cuba.
E-mail: rabreu@facinf.uho.edu.cu

J. Bory Reyes
Departamento de Matemática, Universidad de Oriente, Santiago de Cuba 90500, Cuba.
E-mail: jbory@rect.uo.edu.cu

F. Brackx
Department of Mathematical Analysis, Faculty of Engineering, Ghent University, Galglaan 2, 9000 Gent, Belgium.
E-mail: fb@cage.UGent.be

H. De Schepper
Department of Mathematical Analysis, Faculty of Engineering, Ghent University, Galglaan 2, 9000 Gent, Belgium.
E-mail: hds@cage.UGent.be

F. Sommen
Department of Mathematical Analysis, Faculty of Engineering, Ghent University, Galglaan 2, 9000 Gent, Belgium.
E-mail: fs@cage.UGent.be